J. Fluid Mech. (1967), *vol.* 30, *part* 2, *pp.* 357–369
Printed in Great Britain

On slow transverse motion of a sphere through a rotating fluid

By K. STEWARTSON
Department of Mathematics, University College London

(Received 23 March 1967)

The fluid is assumed to be inviscid and to be confined within two parallel planes, each perpendicular to the axis of rotation. A sphere is set moving, relative to the rotating fluid, in a straight line with uniform velocity and the temporal development of the flow structure examined. It is found that ultimately the flow has different properties inside and outside the cylinder $\mathscr{C}$, circumscribing the sphere and having its generators parallel to the axis of rotation. Inside $\mathscr{C}$ the fluid moves with the sphere as if solid; in early experiments of Taylor (1923) this phenomenon was observed. Outside $\mathscr{C}$ the motion is a two-dimensional potential flow past $\mathscr{C}$ as if it were solid. Then the asymmetry observed by Taylor and predicted in an earlier theory of the author for an unbounded fluid (1953) is not borne out. A partial explanation is offered.

1. Introduction

The experiments carried out by Taylor (1923) on the motion of bodies through rotating fluids have proved a source of interest to many theoretical workers. In one of them, with which we shall be particularly concerned in this paper, the fluid rotated about a vertical axis with angular velocity Ω and a short stubby cylinder was caused to move in a horizontal plane in such a way that, relative to the rotating fluid, it moved in a straight line with uniform velocity U. Taking a to be a characteristic length of the cylinder he found that, when $U/\Omega a \ll 1$ the induced motion of the fluid relative to the rotating axis was cylindrical, i.e. the same in all horizontal planes, and markedly different inside and outside the cylinder $\mathscr{C}$, circumscribing the stubby cylinder and having its generators vertical. Inside $\mathscr{C}$ the fluid was at rest relative to the body, while outside it flowed, asymmetrically, past the cylinder $\mathscr{C}$ as if it were solid. His experiments have recently been repeated by Hide & Ibbetson (1966), who confirm the main details of the motion but who find a weak flow (i.e. $\ll U$) through $\mathscr{C}$ and a flow outside $\mathscr{C}$ which is apparently a little more symmetric than that observed by Taylor. They ascribe the weak flow inside $\mathscr{C}$ to viscous effects.

The cylindrical aspect of the motion can be explained by means of the Taylor–Proudman theorem (Proudman 1916) but no further progress is possible on the basis of the steady equations of inviscid flow. Following on a suggestion of Taylor, Grace (1927) examined the solution of the unsteady inviscid equations in the hope of deducing a steady solution in the limit as $\Omega t \to \infty$. In this investigation the short stubby cylinder was replaced by a sphere and the region occupied by the fluid was unbounded in all directions. His solution was completed by Stewartson (1953), who confirmed Grace's conjecture that the solution became steady almost everywhere as $\Omega t \to \infty$. The principal properties of this flow are that outside $\mathscr{C}$, defined in an equivalent way to the earlier one, the motion is clearly asymmetric and cylindrical, and reconcilable to the observations of Taylor. However, there is a flux of fluid across $\mathscr{C}$, and inside $\mathscr{C}$ the streamlines are arcs of circles on spheres inscribed in $\mathscr{C}$. The disagreement with Taylor's observations in $\mathscr{C}$ is not perhaps surprising as in the experiment the fluid was bounded by a horizontal lid so that the theoretical solution violated the boundary condition there.

An alternative theoretical attack was proposed by Jacobs (1964). Instead of studying the unsteady inviscid equations he considered the steady viscous equations assuming in addition that $R = \Omega a^2/\nu \gg 1$, where ν is the kinematic viscosity. He argued that the non-uniqueness in the solution of the steady inviscid equations is removed by the boundary layers on the two horizontal walls of the container and on the body, together with the shear layer at $\mathscr{C}$. A unique solution is obtained in which the fluid inside $\mathscr{C}$ moves with the body as if solid, while outside $\mathscr{C}$ the motion is in planes perpendicular to the axis of rotation and irrotational. Agreement with Taylor's observations inside is thus found but the asymmetry outside is lost. Jacobs argues and Carrier (1966) supports him that his approach is more meaningful than the unsteady one, the order of limits $t \to \infty$, $\nu \to 0$ being preferred to $\nu \to 0$, $t \to \infty$.

Quite apart from the intrinsic difficulty in his solution for the shear layer at $\mathscr{C}$, which has been discussed elsewhere (Stewartson 1966), Jacobs's strong preference for his limit sequence is not convincing. In the view of the present author both limit sequences are useful and may be regarded as complementary, especially since they lead, as we shall see, to the *same* flow pattern for the same geometry.

Formally the viscous approach suffers from the disadvantage that it is only valid if

$$\frac{U}{a\Omega} \ll R^{-\frac{1}{2}}; \tag{1.1}$$

Jacobs arrived at this condition (apart from a misprint) by considering the 'non-linear and Coriolis accelerations in the Ekman layer on the... obstacle'. Another way to arrive at the range of validity of this approach is to argue that viscous forces can only be expected to be significant if the reaction time of the Ekman boundary layers on the horizontal walls to changes in the flow outside them is much less than the time it takes fluid particles to move into and out of the zone of influence of the body. The first time-scale is the spin-up time-scale

$$\Omega^{-1}\left(\frac{\Omega h^2}{\nu}\right)^{\frac{1}{2}}$$

found by Greenspan & Howard (1963), where $2h$ is the distance between the horizontal walls of the container, and the condition is therefore

$$\Omega^{-1}\left(\frac{\Omega h^2}{\nu}\right)^{\frac{1}{2}} \ll \frac{a}{U}, \tag{1.2}$$

which is equivalent to (1.1). This condition is so restrictive that an experimental test of his theory is a formidable task.

An obvious difficulty in comparing the unsteady inviscid solution with experiment is that the streamlines inside $\mathscr{C}$ lie on spheres, which contradicts the normal experimental situation in which the fluid is bounded by planes. In other problems when this occurs it is possible to reconcile theory and experiment by taking the planes far enough apart but here no such reconciliation is possible because of the cylindrical character of the flow. Hence it is not sufficient, from a theoretical standpoint, to assume the fluid is unbounded and instead the normal velocity must be set equal to zero on each of the bounding planes assumed a distance $2h$ apart.

In this paper we shall investigate the consequences of supposing that $\nu = 0$ and show that as $t \to \infty$ the ultimate flow pattern is identical with that found by Jacobs, but subject only to the weaker restriction $U/\Omega a \ll 1$. This agreement is gratifying and indicates the complementary nature of the two approaches. It is true that the range of validity claimed for the unsteady approach is greater than that claimed for the viscous approach. It may be that the arguments leading to (1.1) and (1.2) are unduly pessimistic and the argument of one referee is the relevant one to use. He writes: 'there is no convincing argument why the right-hand side of (1.2) should represent a time-scale for the generation of relative vorticity due to the presence of the cylinder. There is no reason, provided that the Taylor column forms, why its presence should stretch vortex lines or act in any way to alter the absolute vorticity component parallel to the generators; certainly not on a time-scale a/U'. Attention is largely confined to spherical bodies in the paper, but the method is actually applicable to any smooth body and any permissible value of h. The ultimate flow is in horizontal planes, the fluid is at relative rest inside the cylinder $\mathscr{C}$ circumscribing the body and outside it is in irrotational motion. In addition we shall offer a partial explanation of the observed asymmetry outside $\mathscr{C}$.

2. The statement of the problem

Consider a fluid of constant density ρ rotating about a fixed vertical axis l with uniform angular velocity Ω. At time $t = 0$ a sphere of radius a and centre O is set in motion in a plane at right angles to l in such a way that, relative to axes rotating about l with angular velocity Ω, the centre O describes a straight line with constant speed U. Further $U/a\Omega \ll 1$, so that the motion induced by the sphere is slow relative to the rotating fluid, and viscous effects are neglected, the appropriate condition being the reverse of (1.2). Define a set of rectangular axes $Oxyz$ rotating about l with angular velocity Ω such that Oz is parallel to l and Ox is in the direction of motion of the sphere. Let (u, v, w) be the components of the velocity of the fluid relative to these axes, p the pressure and let

$$\rho P = p - \tfrac{1}{2}\Omega^2 r_l^2,$$

where r_l is the distance of the representative point from l. Then the equations of motion may be reduced to (Stewartson 1953)

$$\left.\begin{aligned} \frac{\partial u}{\partial t} - 2\Omega v + \frac{\partial P}{\partial x} = 0, \quad \frac{\partial v}{\partial t} + 2\Omega u + \frac{\partial P}{\partial y} = 0, \\ \frac{\partial w}{\partial t} + \frac{\partial P}{\partial z} = 0, \quad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0, \end{aligned}\right\} \tag{2.1}$$

squares and products of u, v, w being neglected. The boundary condition at the sphere is that

$$ux+vy+wz = Ux \quad \text{when} \quad x^2+y^2+z^2 = a^2 \quad (t > 0) \tag{2.2a}$$

and at time $t = 0+$ the motion is that classic irrotational flow for an ideal fluid found by setting $\Omega = 0$ in (2.1). We shall suppose here that the fluid is confined between two planes $z = \pm h$ but is otherwise unbounded. Thus an additional boundary condition is that

$$w = 0 \quad \text{at} \quad z = \pm h, \quad \text{i.e.} \quad \partial P/\partial z = 0 \quad \text{at} \quad z = \pm h. \tag{2.2b}$$

For a completely unbounded fluid the full solution of (2.2a) is known (Stewartson 1953) but has the disadvantage that in the limit $2\Omega t = \infty$, $w \not\to 0$ as $z \to \infty$. Hence, even if h/a is large, the condition (2.2) is certain to exert a significant effect on the solution for large enough t. No such difficulty is likely to arise in the x, y directions as the relative fluid velocities die out fairly quickly as x^2+y^2 increases to infinity.

The method of solution for an infinite mass of fluid is to take the Laplace transform of all dependent variables with respect to t and we shall adopt the same procedure here when (2.2b) has to be satisfied. Denoting the parameter of the transform by s and the transform by a bar so that, for example,

$$\bar{u}(x,y,z;s) = \int_0^\infty e^{-st}u(x,y,z,t)\,dt, \tag{2.3}$$

we find, after some reduction, that

$$\left.\begin{aligned} \bar{u} &= -\frac{s}{s^2+4\Omega^2}\frac{\partial \bar{P}_1}{\partial x} - \frac{2\Omega}{s^2+4\Omega^2}\frac{\partial \bar{P}_1}{\partial y}, \quad \bar{v} = \frac{2\Omega}{s^2+4\Omega^2}\frac{\partial \bar{P}_1}{\partial x} - \frac{s}{s^2+4\Omega^2}\frac{\partial \bar{P}_1}{\partial y}, \\ \bar{w} &= -\frac{1}{s}\frac{\partial \bar{P}_1}{\partial z}, \quad \frac{\partial^2 \bar{P}_1}{\partial x^2} + \frac{\partial^2 \bar{P}_1}{\partial y^2} + \frac{s^2+4\Omega^2}{s^2}\frac{\partial^2 \bar{P}_1}{\partial z^2} = 0, \end{aligned}\right\} \tag{2.4}$$

where

$$P_1(x,y,z,t) = P - P_0(x,y,z)\,\delta(t),$$

P_0 being the impulsive pressure at $t = 0$ and $\delta(t)$ the Dirac delta function.

The earlier work indicated, quite clearly, that, except on the sphere and on the axis Oz, the ultimate motion is determined by the behaviour of the barred quantities in (2.4) as $s \to 0$. In these exceptional regions contributions also arose from poles on the imaginary axis of s such that $|s| < 2\Omega$. Since these regions are of zero volume and the unsteady extra motions are essentially free oscillations we can expect that even a very small viscosity would dampen them out and in any case the motions are unmeasurable. Finally, they do not alter the form of the boundary conditions when $s \ll \Omega$ and do not contribute to the force on the sphere. In the present problem we shall therefore disregard them and assume that the motion at large times is solely determined by the structure of the transforms as $s \to 0$. Further we shall focus interest on times t such that $\Omega t \gg 1$, i.e. after many revolutions of the fluid have taken place from the start of the sphere's relative motion. Equivalently therefore $s/\Omega \ll 1$ and (2.4) reduces to

$$\left.\begin{aligned} \bar{u} &= -\frac{s}{4\Omega^2}\frac{\partial \bar{P}_1}{\partial x} - \frac{1}{2\Omega}\frac{\partial \bar{P}_1}{\partial y}, \quad \bar{v} = \frac{1}{2\Omega}\frac{\partial \bar{P}_1}{\partial x} - \frac{s}{4\Omega^2}\frac{\partial \bar{P}_1}{\partial y}, \\ \bar{w} &= -\frac{1}{s}\frac{\partial \bar{P}_1}{\partial z}, \quad \frac{\partial^2 \bar{P}_1}{\partial x^2} + \frac{\partial^2 \bar{P}_1}{\partial y^2} + \frac{4\Omega^2}{s^2}\frac{\partial^2 \bar{P}_1}{\partial z^2} = 0, \end{aligned}\right\} \tag{2.5}$$

and the boundary condition on the sphere to

$$\frac{s}{4\Omega^2}\left[-x\frac{\partial \bar{P}_1}{\partial x} - y\frac{\partial \bar{P}_1}{\partial y}\right] + \frac{1}{2\Omega}\left[-x\frac{\partial \bar{P}_1}{\partial y} + y\frac{\partial \bar{P}_1}{\partial x}\right] - \frac{z}{s}\frac{\partial \bar{P}_1}{\partial z} = \frac{xU}{s}, \tag{2.6}$$

when $x^2+y^2+z^2 = a^2$. From the differential equation we can write

$$\bar{P}_1 = \bar{Q}(r,\theta,\xi)\frac{2\Omega U a}{s}, \tag{2.7}$$

where $$x = ar\cos\theta, \quad y = ar\sin\theta, \quad \text{and} \quad \xi = sz/2\Omega a. \tag{2.8}$$

In terms of $\bar{Q}$, $$\frac{\partial^2 \bar{Q}}{\partial r^2} + \frac{1}{r}\frac{\partial \bar{Q}}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \bar{Q}}{\partial \theta^2} + \frac{\partial^2 \bar{Q}}{\partial \xi^2} = 0, \tag{2.9}$$

$$\frac{sr}{2\Omega}\frac{\partial \bar{Q}}{\partial r} + \frac{\partial \bar{Q}}{\partial \theta} + (1-r^2)^{\frac{1}{2}}\operatorname{sgn}\xi\frac{\partial \bar{Q}}{\partial \xi} = -r\cos\theta, \tag{2.10a}$$

when $$s^2r^2 + 4\Omega^2\xi^2 = s^2 \tag{2.10b}$$

and $$\frac{\partial \bar{Q}}{\partial \xi} = 0 \quad \text{when} \quad \xi = \pm\frac{sh}{2\Omega a} = \pm H. \tag{2.11}$$

Since $2\Omega \gg s$ (2.10*b*) can be replaced by $\xi = 0$ and the first term of (2.10*a*) may be neglected.

We are particularly interested, in this paper, in the solution when $h/a \gg 1$ so that the diameter of the sphere is small compared with the distance between the top and base plates of the fluid container. When this condition is satisfied we see from (2.11) that the fluid is effectively infinite when $H \gg 1$, i.e. $2\Omega t \ll h/a$, but that if $H \ll 1$, i.e. $2\Omega t \gg h/a$, the horizontal boundary planes exert a decisive influence on the motion. In the next two sections we shall consider these two temporal regions in more detail.

3. Solution characteristics when $1 \ll 2\Omega t \ll h/a$.

For this range of times we may neglect s in comparison with 2Ω and take $sh \gg 2\Omega a$. The governing equation and boundary conditions are the simplified versions (2.9)–(2.11) and we proceed by writing

$$\bar{Q} = \mathscr{R}e^{i\theta}S(r,\xi), \tag{3.1}$$

whence S satisfies $$\frac{\partial^2 S}{\partial r^2} + \frac{1}{r}\frac{\partial S}{\partial r} - \frac{S}{r^2} + \frac{\partial^2 S}{\partial \xi^2} = 0, \tag{3.2}$$

with boundary conditions

$$iS + \sqrt{(1-r^2)}\,\mathrm{sgn}\,\xi \frac{\partial S}{\partial \xi} = -r \quad \text{when} \quad \xi = 0, \quad r < 1; \tag{3.3}$$

$$\frac{\partial S}{\partial \xi} = 0 \quad \text{when} \quad \xi = 0, \quad r > 1 \quad \text{and when} \quad \xi = H \tag{3.4}$$

by symmetry, where $H \gg 1$. The solution in $0 < \xi < H$ may formally be expressed as an integral

$$S = \int_0^\infty J_1(kr)\frac{\cosh k(H-\xi)}{\sinh kH} A(k)\,dk, \tag{3.5}$$

where $A(k)$ is a function to be found and J_1 is a Bessel function. This solution satisfies the boundary condition at $\xi = H$ automatically, can be extended into the region $\xi < 0$ by symmetry, and satisfies the conditions at $\xi = 0$ if

$$\int_0^\infty kA(k)\,J_1(kr)\,dk = 0 \quad (r > 1), \tag{3.6a}$$

and

$$i\int_0^\infty A(k)J_1(kr)\coth kH\,dk - (1-r^2)^{\frac{1}{2}}\int_0^\infty kA(k)\,J_1(kr)\,dk = -r \quad (r<1). \tag{3.6b}$$

The solution of this dual set of integral equations can be expressed, when $H \gg 1$, as a series of Bessel functions. First take $H = \infty$, when (3.6*b*) reduces to

$$i\int_0^\infty A(k)\,J_1(kr)\,dk - (1-r^2)^{\frac{1}{2}}\int_0^\infty kA(k)\,J_1(kr)\,dk = -r \quad (r<1); \tag{3.7}$$

reference to Watson (1944) now shows that the appropriate solution of (3.6*a*), (3.7) is

$$A(k) = B\left(\frac{2\pi}{k}\right)^{\frac{1}{2}} J_{\frac{3}{2}}(k), \tag{3.8}$$

for, on substituting into the left-hand sides we find that (3.6*a*) is automatically satisfied while (3.7) reduces to

$$\frac{\pi r}{2}Bi - (1-r)^{\frac{1}{2}}\frac{2r}{(1-r^2)^{\frac{1}{2}}}B = -r.$$

Hence

$$B = \frac{2}{4-\pi i}$$

and the corresponding values of S when $\xi = 0$ are

$$\frac{\pi r}{4-\pi i} \quad \text{if} \quad r<1 \quad \text{and} \quad \frac{2r}{4-\pi i}\left(\sin^{-1}\frac{1}{r} - \frac{(r^2-1)^{\frac{1}{2}}}{r^2}\right) \quad \text{if} \quad r>1. \tag{3.9}$$

It follows that when z is finite

$$\bar{P}_1 = \frac{2U\Omega\pi(4x-\pi y)}{(16+\pi^2)\,s} \quad \text{if} \quad r<1 \tag{3.10}$$

with an equivalent result for $r > 1$. Thus for values of t satisfying $1 \ll 2\Omega t \ll h/a$,

$$P = \frac{2\Omega U\pi(4x-\pi y)}{16+\pi^2} + \dots \quad \text{if} \quad x^2+y^2 \leqslant a^2$$

and

$$P = \frac{4U\Omega(4x-\pi y)}{16+\pi^2}\left(\sin^{-1}\frac{a}{(x^2+y^2)^{\frac{1}{2}}} - \frac{a(x^2+y^2-a^2)^{\frac{1}{2}}}{x^2+y^2}\right) \quad \text{if} \quad x^2+y^2 \geqslant a^2 \tag{3.11}$$

provided $z/a \sim 1$. These results agree with those obtained earlier from the inverse of the full solution. The previous method was more complicated, however, and the number of different body shapes that could be considered using it is severely restricted. Greater scope is allowed by the present method, since all three-dimensional bodies are reduced to equivalent disks. For example, if the body is symmetrical about the z-axis, the method developed by Collins (1961) in which S is expressed as an integral of sources distributed along the imaginary z-axis enables the evaluation of S to be reduced to the solution of an integral equation for the source function. We shall not pursue this point, however, because it is not germane to our main purpose of investigating the effect of the plane boundaries.

When H is large but not infinite the solution (3.8) may be generalized by expanding $\coth kH$ as a series of exponentials,

$$\coth kH = 1+2\sum_{n=1}^{\infty} e^{-2nkH}$$

and re-writing (3.6*b*) as

$$i\int_0^\infty A(k)\,J_1(kr)\,dk-(1-r^2)^{\frac{1}{2}}\int_0^\infty kA(k)\,J_1(kr)\,dk = -r-2i\sum_{n=1}^{\infty}\int_0^\infty e^{-2nkH}A(k)\,J_1(kr)\,dk, \qquad (3.12)$$

if $r < 1$. After further reference to Watson (1944) it may be seen that $A(k)$ can be written as a series of Bessel functions, viz:

$$A(k) = \left(\frac{2\pi}{k}\right)^{\frac{1}{2}}\left[B_1 J_{\frac{3}{2}}(k)+\frac{B_2}{k}J_{\frac{5}{2}}(k)+\ldots\right] \qquad (3.13)$$

where, to anticipate, B_1, B_2 are independent of k, $B_1 = O(1)$ and $B_2 = O(H^{-5})$. On substituting into (3.12) and neglecting terms $O(H^{-6})$ we find that

$$B_1 = \frac{2}{4-\pi i}+\frac{2i\zeta(3)}{3H^3(4-\pi i)^2}+O(H^{-5}), \qquad (3.14a)$$

$$B_2 = \frac{2i\zeta(5)}{(4-\pi i)\,(32-3\pi i)H^5}+\ldots, \qquad (3.14b)$$

where $\zeta(\alpha)$ is the Riemann zeta function defined by

$$\zeta(\alpha) = \sum_{n=1}^{\infty}\frac{1}{n^\alpha}.$$

In addition, if $0 < r < 1, \xi = 0$,

$$\bar{Q} = \mathscr{R}\left\{\frac{\pi r}{2}B_1 e^{i\theta}+\frac{\pi r}{4}\left(1-\frac{3r^2}{4}\right)B_2 e^{i\theta}+\ldots\right\}$$

and, inverting the Laplace transform to find P_1 at finite values of z,

$$P_1 = 2U\Omega\,\mathscr{R}\left[\frac{\pi a r e^{i\theta}}{2(4-\pi i)}+\frac{\pi a r e^{i\theta}\zeta(3)}{2(4-\pi i)^2\,3!}\left(\frac{2\Omega a t}{h}\right)^3+\ldots\right]. \qquad (3.15)$$

Thus as $2\Omega\, at/h$ increases from very small values the cylindrical character of the motion is preserved but the asymmetry is slowly modified. An indication of how this occurs is obtained from the force on the sphere due to the relative motion of the fluid past it. This has components $(F_x, F_y, 0)$, where

$$F_x = -\iint \rho P \frac{x}{a} dS, \quad F_y = -\iint \rho P \frac{y}{a} dS, \tag{3.16}$$

the integrals being taken over the sphere. Hence

$$\bar{F}_x - i\bar{F}_y = -2\pi a^2 \frac{2U\Omega a}{s} \int_0^1 \frac{r^2}{(1-r^2)^{\frac{1}{2}}} S(r,0)\, dr$$
$$= -\frac{4\pi a^3 U\Omega}{s}\left[\frac{\pi B_1}{3} + \frac{\pi B_2}{15} + \ldots\right] \tag{3.17}$$

and, inverting,

$$F_x = -4\pi a^3 U\Omega \left[\frac{8\pi}{3(\pi^2+16)} - \frac{8\pi^2 \zeta(3)}{27(16+\pi^2)^2}\left(\frac{2\Omega at}{h}\right)^3 + \ldots\right],$$
$$F_y = 4\pi a^3 U\Omega \left[\frac{2\pi^2}{3(\pi^2+16)} + \frac{\pi(16-\pi^2)}{27(16+\pi^2)^2}\left(\frac{2\Omega at}{h}\right)^3 + \ldots\right]. \tag{3.18}$$

The direction of the force on the sphere thus moves round towards the transverse or y direction and since the pressure term P_1 acts like a streamfunction this indicates that the flow is becoming more symmetrical. In the next section we shall see that when $2\Omega at/h \gg 1$ the motion is entirely symmetric.

4. The motion when $2\Omega at/h \gg 1$

The solution of the dual integral equations (3.6) is not known for general values of H but it is possible to make some progress with the determination of S when H is small. Since S satisfies (3.2) it follows that, apart from the neighbourhood of certain surfaces, we can expect $\partial/\partial r$ and $\partial/\partial \xi$ to be of the same order of magnitude. Hence, when $H \ll 1$, $\partial S/\partial \xi$ will be small on $\xi = 0$ because it vanishes on $\xi = H$. The boundary conditions then reduce to

$$\partial S/\partial \xi = 0 \quad \text{at} \quad \xi = H \quad \text{and at} \quad \xi = 0, \quad r > 1 \tag{4.1}$$

by symmetry while

$$S = ir \tag{4.2}$$

if $r < 1$, $\xi = 0$. A formal solution of (3.2) can now be written down in which $S = ir$ for $0 \leqslant \xi \leqslant H$, $0 \leqslant r \leqslant 1$ and

$$S = i/r, \quad r > 1, \quad 0 \leqslant \xi \leqslant H, \tag{4.3}$$

the arbitrary constant in $r > 1$ being fixed by requiring S to be continuous at $r = 1$.† Although (4.2), (4.3) is an exact solution of (3.2) almost everywhere (for all $H > 0$) it has a discontinuous normal derivative at $r = 1$ which it must be possible to smooth out in an arbitrarily thin 'shear layer' before the solution can be accepted as relevant to our problem. We shall now show this is possible as $H \to 0$ and for this purpose we consider the neighbourhood of the cylinder $r = 1$, writing

$$\xi = H\tau, \quad r = 1 + H\sigma, \quad S = ir + HT(\sigma, \tau). \tag{4.4}$$

It is supposed here that $0 \leqslant \tau \leqslant 1$ and that although σ may become large in the region of interest $H\sigma$ remains small. In fact it will appear that the region where

† The reader is referred to the footnote on p. 367 for the consequences of a discontinuity in S at $r = 1$.

$|\sigma| \sim H^{-\frac{1}{3}}$ is of particular importance for determining the properties of T, but the criterion $|\sigma H| \ll 1$ is still satisfied when $H \ll 1$. The equation satisfied by T reduces to

$$\frac{\partial^2 T}{\partial \sigma^2} + \frac{\partial^2 T}{\partial \tau^2} = 1 \tag{4.5}$$

and the boundary conditions to

$$\frac{\partial T}{\partial \tau} = 0 \quad \text{at} \quad \tau = 1 \quad \text{and at} \quad \tau = 0 \quad \text{for} \quad \sigma > 0, \tag{4.6a}$$

$$iT + \left(-\frac{2\sigma}{H}\right)^{\frac{1}{2}} \frac{\partial T}{\partial \tau} = 0 \quad \text{at} \quad \tau = 0, \quad \sigma < 0, \tag{4.6b}$$

$$T \to 0 \quad \text{as} \quad \sigma \to -\infty \quad \text{and} \quad T \to -2i + C \quad \text{as} \quad \sigma \to +\infty, \tag{4.6c}$$

where C is independent of σ and has to be found. It turns out that $|C| \gg 1$ [see (4.19) below], that $|T|$ increases from zero to a value approximately equal to $|C|$ as σ increases from $-\infty$ to values $O(H^{-\frac{1}{3}})$ and that subsequently the variation of $|T|$ is relatively small so long as $|\sigma| = O(1)$.

We observe that if

$$\frac{\partial T}{\partial \tau} = f(\sigma) \tag{4.7}$$

when $\sigma < 0$, and $\tau = 0$, then by integrating (4.5) with respect to τ from 0 to 1

$$\frac{\partial^2}{\partial \sigma^2} \int_0^1 T\, d\tau \begin{array}{ll} = 0 & (\sigma > 0) \\ = f(\sigma) & (\sigma < 0); \end{array}$$

whence, on integrating with respect to σ from $-\infty$ to ∞ and using the boundary condition (4.6c),

$$\int_{-\infty}^{0} f(\sigma)\, d\sigma = -2i. \tag{4.8}$$

Now take σ negative but finite so that $T \approx C$. It then follows from (4.6b) that

$$f(\sigma) = \left.\frac{\partial T}{\partial \tau}\right|_{\tau=0} = -iC\left(\frac{H}{-2\sigma}\right)^{\frac{1}{2}}; \tag{4.9}$$

when $-\sigma$ is large but still finite the appropriate form for T is a quadratic in τ whose coefficients are functions of σ, the other terms in the solutions being either exponentially small or algebraic and negligible. Thus

$$T = -\frac{iC}{2}\left(\frac{H}{-2\sigma}\right)^{\frac{1}{2}} (2\tau - \tau^2) + C + \dots. \tag{4.10}$$

This form for T does not formally satisfy (4.5), the leading error term needing an extra term $-\frac{4}{3} iC\sigma^2 (H/-2\sigma)^{\frac{1}{2}}$ in (4.10) to cancel it. In confirmation of the remark just made this new term is algebraic and negligible when σ is finite but clearly can no longer be neglected when $|\sigma| \sim H^{-\frac{1}{3}}$ as it is then of the same order of magnitude as C. The appropriate form for T is then suggested by (4.10), which we generalize to

$$T(\sigma, \tau) = T_0(\sigma) + \tfrac{1}{2} T_1(\sigma)(2\tau - \tau^2) + \tfrac{1}{24} T_2(\sigma)(8\tau - 4\tau^3 + \tau^4) + \dots \tag{4.11}$$

and now we suppose that $-\sigma H^{\frac{1}{3}} \sim 1$. The form of the nth term of the series in

(4.11) is $T_{n-1}(\sigma)f_{n-1}(\tau)$, where $f''_n(\tau) = -f_{n-1}(\tau), f'_n(1) = 0, f_n(0) = 0$. On substituting into the differential equation (4.5)

$$T''_0(\sigma) = T_1(\sigma), \quad T''_1(\sigma) = T_2(\sigma) \text{ etc.,} \tag{4.12}$$

while from the boundary condition (4.6*b*)

$$iT_0(\sigma) + \left(\frac{-2\sigma}{H}\right)^{\frac{1}{2}} (T_1(\sigma) + \tfrac{1}{3}T_2(\sigma) + \ldots) = 0. \tag{4.13}$$

Now from (4.12), $T_0 \sim H^{-\frac{2}{3}}T_1$, $T_2 \sim H^{\frac{2}{3}}T_1$, etc., and hence T_2 may safely be neglected in (4.13), which reduces to

$$T''_0(\sigma) + i\left(\frac{H}{-2\sigma}\right)^{\frac{1}{2}} T_0(\sigma) = 0. \tag{4.14}$$

It may now be confirmed that $\sigma H^{\frac{1}{3}}$ is the relevant variable in $T_0(\sigma)$ and the boundary conditions require that $T_0(\sigma) \to 0$ as $-\sigma H^{\frac{1}{3}} \to \infty$ while $T_0(\sigma) \to C$ as $-\sigma H^{\frac{1}{3}} \to 0$. The solution of (4.14) satisfying these conditions is

$$T_0(\sigma) = C\frac{Ai'(\eta)}{Ai'(0)}, \tag{4.15}$$

where $\eta = e^{-\frac{1}{6}\pi i}(-2\sigma H^{\frac{1}{3}})^{\frac{1}{2}}$ and Ai is the Airy function. It is noted that

$$Ai(0) = \frac{(-\frac{2}{3})!\,3^{-\frac{1}{6}}}{2\pi}, \quad Ai'(0) = -\frac{1}{2\pi}3^{\frac{1}{6}}(-\tfrac{1}{3})! \quad \text{and} \quad Ai'(\eta) \sim \frac{1}{2\sqrt{\pi}}\eta^{\frac{1}{4}}e^{-\frac{2}{3}\eta^{\frac{3}{2}}}, \tag{4.16}$$

as $\eta \to \infty$. From (4.13) and (4.19)

$$f(\sigma) = -i\left(\frac{H}{-2\sigma}\right)^{\frac{1}{2}} C\frac{Ai'(\eta)}{Ai'(0)} \tag{4.17}$$

and from (4.8)

$$2i = \frac{-iCe^{\frac{1}{6}\pi i}H^{\frac{1}{3}}}{Ai'(0)}\int_0^\infty Ai'(\eta)\,d\eta$$

so that

$$C = 2\frac{(-\frac{1}{3})!}{(-\frac{2}{3})!}e^{-\frac{1}{6}\pi i}\left(\frac{3}{H}\right)^{\frac{1}{3}}. \tag{4.18}$$

It remains now to verify that when $|\sigma| \sim 1$ the changes in T are small in comparison with C. For then $\partial T/\partial\tau$ is known on $\tau = 0$ being either zero or $\sim H^{\frac{1}{6}}$. The determination of T is now a straightforward problem, the solution being expressed as a Fourier integral and it is easy to verify that the change in T is also $\sim H^{\frac{1}{6}}$ and small in comparison with C. As $\sigma \to +\infty$, T increases linearly with σ but this does not affect our argument. It is concluded that when $H \ll 1$ the discontinuity at $r = 1$ in the normal derivative of the simple solution in (4.2), (4.3) can be smoothed out in a region of width $\sim H^{-\frac{1}{3}}$ in σ and therefore $\sim H^{\frac{2}{3}}$ in r. The consequent change in T in this region $\sim H^{-\frac{1}{3}}$ and of $S \sim H^{\frac{2}{3}}$. So far as we

can test therefore (4.2), (4.3) is the correct limit of S and $H \to 0$ and, with confidence, we can now proceed with the interpretation of the solution in terms of t.†
From (2.7) and (3.1)

$$\left.\begin{aligned} \bar{P}_1 &= \frac{2\Omega Ua}{s}\mathscr{R}ire^{i\theta} \quad \text{if} \quad r < 1, \\ &= \frac{2\Omega Ua}{s}\mathscr{R}\frac{ie^{i\theta}}{r} \quad \text{if} \quad r > 1, \end{aligned}\right\} \tag{4.19}$$

as $H \to 0$, so that

$$\left.\begin{aligned} P_1 &\to -2\Omega Uy \quad \text{if} \quad x^2+y^2 \leqslant a^2, \\ \text{and} \qquad &\to -\frac{2\Omega Uya^2}{x^2+y^2} \quad \text{if} \quad x^2+y^2 \geqslant a^2, \end{aligned}\right\} \tag{4.20}$$

as $2\Omega at/h \to \infty$. Hence

$$u \to U, \quad v \to 0, \quad w \to 0 \quad \text{if} \quad x^2+y^2 \leqslant a,$$

while

$$u \to \frac{U(x^2-y^2)a^2}{(x^2+y^2)^2}, \quad v \to \frac{2Uxya^2}{(x^2+y^2)^2}, \quad w \to 0 \quad \text{if} \quad x^2+y^2 > a^2 \tag{4.21}$$

as $2\Omega at/h \to \infty$.

This solution indicates that in the ultimate motion the cylinder $\mathscr{C}$ of fluid circumscribing the sphere and having its generators parallel to the axis of rotation l is carried along with the sphere as if solid (i.e. no relative motion) while outside $\mathscr{C}$ the fluid describes a symmetric irrotational motion past it, the whole taking place in planes perpendicular to l. It is noted that this solution is the same as that of Jacobs (1964), so that the limits $\nu \to 0$, $t \to \infty$ are, in some sense, commutative. Further for a real fluid the inviscid shear layer we have been discussing will ultimately be modified either by the action of viscosity or by the non-linear terms in the governing equations, or both, and will presumably remain thin, but of finite thickness as $\Omega t \to \infty$. The solution is capable of generalization to any finite body in a straightforward manner. There is no need to reproduce the argument, it being sufficient to point out that the flow pattern described earlier in this paragraph is also applicable to the general body if $\mathscr{C}$ is re-defined to circumscribe the body and that the structure of the flow near $\mathscr{C}$ is virtually identical with that for the sphere. A further generalization which one can make is to the case when h and a are of the same order. Now §3 is irrelevant and it is no longer legitimate to approximate the sphere boundary by $\xi = 0$ as is done in (3.3) and (4.6b) but we must take it to be

$$\xi = \pm\frac{s}{2\Omega}(1-r^2)^{\frac{1}{2}}.$$

† This result implies that

$$[S] = \left[\frac{\partial S}{\partial r}\right] O(H^{\frac{1}{2}})$$

where $[S]$ denotes the leap in S across the shear layer near $r = 1$. Hence a discontinuity in S of order one is unacceptable since the limit $H \to 0$ must be associated with a singularity in $\partial S/\partial r$ as $r \to 1+$.

However, the solution in (4.2), (4.3) goes through as before when $2\Omega \gg s$ and so does the study of the neighbourhood of $\mathscr{C}$ because the sphere boundary in terms of τ is here

$$\tau \approx \frac{a}{h}(-2\sigma H)^{\frac{1}{2}}$$

and is small when $\sigma \sim H^{-\frac{1}{3}}$. It follows therefore that for all smooth bodies and all allowed values of h the final motion is the same—inside $\mathscr{C}$ the fluid is at rest relative to the body while outside it is in irrotational motion past $\mathscr{C}$.

When the solution is compared with Taylor's experiments the agreement is good so far as the motion inside $\mathscr{C}$ is concerned, for he too found that the fluid inside $\mathscr{C}$ was carried along with the sphere as if solid. On the other hand the flow observed by Taylor outside $\mathscr{C}$ was highly asymmetric, which is not confirmed here. It is noted, in parenthesis, that the flow pattern found in §3, valid when $1 \ll 2\Omega t \ll h/a$ *is* highly asymmetric outside $\mathscr{C}$ and can be reconciled there with experiment. A possible explanation, or perhaps a mitigation, of the discrepancy between theory and experiment outside $\mathscr{C}$ when $2\Omega at \gg h$ may be found by studying the approach to the limit flow as $2\Omega at/h \to \infty$.

If $r > 1$ the major contribution to S arises from C, which essentially implies that outside the immediate neighbourhood of $r = 1$

$$S = \frac{1}{r}(i + CH) + \dots. \tag{4.22}$$

Hence

$$P = \frac{2\Omega U a}{r}\left[-\sin\theta + c\cos(\theta - \tfrac{1}{6}\pi)\left(\frac{h}{2\Omega at}\right)^{\frac{2}{3}} + \dots\right] \tag{4.23}$$

as $2\Omega at/h \to \infty$ where c is a constant. Thus the asymmetry decays only algebraically. On the other hand, if $r < 1$, $S = ir$ except with a distance $\sim H^{\frac{2}{3}}$ of $r = 1$ and the difference decays exponentially as $(1-r)H^{-\frac{2}{3}}$ increases. Formally therefore we must expect that P approaches its limiting value exponentially except near $r = 1$ although it has not been possible to show this rigorously. The transition region near C is, also formally, of thickness $\sim (2\Omega at/h)^{-\frac{2}{3}}$ on the inner side and of thickness $\sim (2\Omega at/h)^{-1}$ on the outer side. These results suggest that the asymmetry persists for a longer time outside $\mathscr{C}$ than inside $\mathscr{C}$.

To conclude the paper we compute the force on the sphere at large times. Using (3.17) we have

$$\begin{aligned}\overline{F}_x - i\overline{F}_y &= -\frac{4\pi a^3 U\Omega}{s}\left[\int_0^1 \frac{ir^3\,dr}{(1-r^2)^{\frac{1}{2}}} + H^2\int_{-\infty}^0 \frac{T_0(\sigma)\,d\sigma}{(-2\sigma H)^{\frac{1}{2}}} + \dots\right] \\ &= -\frac{4\pi a^3 U\Omega}{s}\left[\frac{2i}{3} + 2H + \dots\right] \end{aligned} \tag{4.24}$$

when $H \ll 1$. Thus

$$F_x - iF_y = -\frac{8\pi a^3 U\Omega i}{3}\left(1 + o\left(\frac{h}{2\Omega at}\right)\right) \tag{4.25}$$

when t is large: it is likely that the leading error term in (4.25) is in fact $O(h/2\Omega at)^{\frac{2}{3}}$.

REFERENCES

CARRIER, G. F. 1966 *Proc. XIth intern. Congr. Appl. Mech.* (*Munich* 1964), p. 69.
COLLINS, W. D. 1961 *Qt. J. Mech. Appl. Math.* **14**, 101.
GRACE, S. F. 1927 *Proc. Roy. Soc.* A **113**, 213.
GREENSPAN, H. P. & HOWARD, L. N. 1963 *J. Fluid Mech.* **17**, 385.
HIDE, R. & IBBETSON, A. 1966 *Icarus*, **5**, 279.
JACOBS, S. J. 1964 *J. Fluid Mech.* **20**, 581.
PROUDMAN, J. 1916 *Proc. Roy. Soc.* A **92**, 408.
STEWARTSON, K. 1953 *Qt. J. Mech. Appl. Math.* **6**, 141.
STEWARTSON, K. 1966 *J. Fluid Mech.* **26**, 131.
TAYLOR, G. I. 1923 *Proc. Roy. Soc.* A **104**, 213.
WATSON, G. N. 1944 *The Theory of Bessel Functions*, pp. 398–404. Cambridge University Press.